Problem 5A,1
Suppose $(X,\mathcal{S}),(Y,\mathcal{T})$ are measurable spaces.Prove that if A is a nonempty set of x and B is a nonempty set of Y such that $A\times B\in\mathcal{S}\otimes\mathcal{T}$, then $A\in\mathcal{S},B\in\mathcal{T}$.

Proof. Consider the characteristic function of $A\times B\in\mathcal{S}\otimes\mathcal{T}$. This function is measurable so by 5.9 the cross section in both direction is measurable, which means $A\in\mathcal{S},B\in\mathcal{T}$. □

Problem 5A,4
Suppose $(X,\mathcal{S}),(Y,\mathcal{T})$ are measurable spaces.Prove that if $f:X\to\mathbb{R}$ is $\mathcal{S}$-measurable and $g:Y\to\mathbb{R}$ is $\mathcal{T}$-measurable and $h:X\times Y\to\mathbb{R}$ is defined to be $h(x,y)=f(x)g(y)$, then h is $\mathcal{S}\otimes\mathcal{T}$-measurable.

Proof. Obviously the result holds if both f,g are characteristic function so moreover if both f,g are simple function. Then we can use simple function to approximate general f,g (2.89), so the result also holds for this case. □

Problem 5A,8
Suppose μ is a measure on measurable space $(X,\mathcal{S})$,prove the following are equivalent:

- μ is σ-finite.
- There exists an increasing sequence $X_1\subset X_2\subset\ldots$ of sets in $\mathcal{S}$ such that $X=\bigcup_{k=1}^{\infty}X_k$ and $\mu(X_k)<\infty$ for every $k\in\mathbb{Z}^+$.
- There exists an disjoint sequence $X_1,X_2,\ldots$ of sets in $\mathcal{S}$ such that $X=\bigcup_{k=1}^{\infty}X_k$ and $\mu(X_k)<\infty$ for every $k\in\mathbb{Z}^+$.

Proof. $2\Rightarrow 3$ is by set $Y_k=X_k-X_{k-1}$ then Y_k is a disjoint sequence. $3\Rightarrow 2$ is by set $Y_k=\bigcup_{j=1}^{k}X_j$. $2\Rightarrow 1$ and $3\Rightarrow 1$ is obvious. $1\Rightarrow 2$ is just by set $Y_k=\bigcup_{j=1}^{k}X_j$. □

Problem 5A,9
Suppose μ,ν is σ-finite measure. Prove that $\mu\times\nu$ is also σ-finite.

Proof. Let $X=\bigcup_{k=1}^{\infty}X_k$ and $Y=\bigcup_{k=1}^{\infty}Y_k$ be the correspoding decompostions of μ,ν,which are also increasing sequences. Then $X\times Y=\bigcup_{k=1}^{\infty}X_k\times Y_k$ satisfies $\mu\times\nu(X_k\times Y_k)=\mu(X_k)\nu(Y_k)<\infty$. □

Problem 5B,1

- Let λ be the Lebesgue meausre on $[0,1]$. Show that

$$\int_{[0,1]}\int_{[0,1}\frac{x^2-y^2}{(x^2+y^2)^2}d\lambda(x)d\lambda(y)=-\frac{\pi}{4}$$

$$\int_{[0,1]}\int_{[0,1}\frac{x^2-y^2}{(x^2+y^2)^2}d\lambda(y)d\lambda(x)=\frac{\pi}{4}$$

- Explain why (a) violates neither Tonelli's theorem nor Fubini's theorem.

Proof. We only calculate the second formula. Write

$$\frac{x^2-y^2}{(x^2+y^2)^2}=\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}=\frac{1}{x^2+y^2}-\frac{2y^2}{(x^2+y^2)^2}$$

Then we have
$$\int_{[0,1]} \frac{2y^2}{(x^2+y^2)^2}dy = \int_{[0,1]} -yd\frac{1}{x^2+y^2} = -\frac{1}{x^2+1} + \int_{[0,1]} \frac{dy}{x^2+y^2}$$
so
$$\int_{[0,1]} \int_{[0,1} \frac{x^2-y^2}{(x^2+y^2)^2} d\lambda(x) d\lambda(y) = \int_{[0,1]} \frac{1}{x^2+1}dx = \frac{\pi}{4}$$

The function $\frac{x^2-y^2}{(x^2+y^2)^2}$ is not positive so doesn't satisfy the assumption of Tonelli's theorem. Also the function is not integrable on $[0,1] \times [0,1]$ so doesn't satisfy the assumption of Fubini's theorem. □

Problem 5B,4
Suppose $(X, \mathcal{S})$ is a measurable space and $f : X \to \mathbb{R}$ is a funciton. Let $graph(X) \subset X \times \mathbb{R}$ denote the graph of f:
$$graph(f) = \{(x, f(x)) | x \in X\}$$
Let $\mathcal{B}$ denote the Borel σ-algebra of $\mathbb{R}$. Prove that $graph(f) \in \mathcal{S} \times \mathcal{B}$ if $f : X \to \mathbb{R}$ is a $\mathcal{S}$-measurable funciton.

Proof. If $f : X \to \mathbb{R}$ is a $\mathcal{S}$-measurable funciton, take
$$E_k = \bigcup_{j=1}^{2^k k-1} f^{-1}([\frac{j}{2^k}, \frac{j+1}{2^k})) \times [\frac{j}{2^k}, \frac{j+1}{2^k}], F_k = f^{-1}([2^k k, \infty]) \times [k2^k, \infty), G_k = E_k \bigcup F_k.$$
Then we have G_k is $\mathcal{S} \times \mathcal{B}$-measurable and $\lim_{k\to\infty} G_k = graph(f)$, which shows $graph(f) \in \mathcal{S} \times \mathcal{B}$. □